Základní setup

- $d, m \in \mathbb{N}$ (budeme hledat zobrazení $\varphi : \mathbb{R}^d \to \mathbb{R}^m$)
- $G \subset \mathbb{R}^{d+m}$ otevřená (souřadnice $\mathbb{R}^{d+m}$ budeme značit $(x, u) = (x_1, \dots, x_d, u_1, \dots u_m)$, případně x, y, z resp. u, v, w)
- $(a, b) \in G$, kde $(a, b) = (a_1, \dots, a_d, b_1, \dots, b_m)$ (bod, který je řešením rovnic a na jehož okolí hledáme další řešení)
- $F : G \to \mathbb{R}^m$, tedy $F = (F_1, \dots, F_m)$, $F_i : G \to \mathbb{R}$
- $F(a, b) = 0$ $(= (0, \dots, 0) \in \mathbb{R}^m)$, můžeme zapsat i jako soustavu

$$F_1(a_1, \dots, a_d, b_1, \dots, b_m) = 0$$
$$\vdots$$
$$F_m(a_1, \dots, a_d, b_1, \dots, b_m) = 0$$

Chceme $\varphi(a) = b$, $F(x, \varphi(x)) = 0$, nebo jako soustavu

$$F_1(x_1, \dots, x_d, \varphi_1(x_1, \dots, x_d), \dots, \varphi_m(x_1, \dots, x_d)) = 0$$
$$\vdots$$
$$F_m(x_1, \dots, x_d, \varphi_1(x_1, \dots, x_d), \dots, \varphi_m(x_1, \dots, x_d)) = 0$$

Věta (o implicitním zobrazení)

Nechť $G \subset \mathbb{R}^{d+m}$ je otevřená, $F : G \to \mathbb{R}^m$, $F \in C^k(G)$ pro nějaké $k \in \mathbb{N}$, $(a, b) \in G \subset \mathbb{R}^{d+m}$. Předpokládejme, že $F(a, b) = 0$ a že platí

$$\det \begin{pmatrix} \frac{\partial F_1}{\partial u_1}(a, b) & \frac{\partial F_1}{\partial u_2}(a, b) & \dots & \frac{\partial F_1}{\partial u_m}(a, b) \\ \frac{\partial F_2}{\partial u_1}(a, b) & \frac{\partial F_2}{\partial u_2}(a, b) & \dots & \frac{\partial F_2}{\partial u_m}(a, b) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial u_1}(a, b) & \frac{\partial F_m}{\partial u_2}(a, b) & \dots & \frac{\partial F_m}{\partial u_m}(a, b) \end{pmatrix} \neq 0.$$

Potom existují $\delta, \Delta > 0$ taková, že

- *pro každé $x \in U(a, \delta)$ existuje právě jedno $y \in U(b, \Delta)$ takové, že $F(x, u) = 0$,*
- *je-li φ zobrazení přiřazující bodu x bod y jako výše, je toto zobrazení $C^k(U(a, \delta))$.*

pro $d = m = 1$

Věta (o implicitní funkci)

Nechť $G \subset \mathbb{R}^2$ *je otevřená,* $F : G \to \mathbb{R}$*,* $F \in C^k(G)$ *pro nějaké* $k \in \mathbb{N}$*,* $(a, b) \in G \subset \mathbb{R}^2$*. Předpokládejme, že*

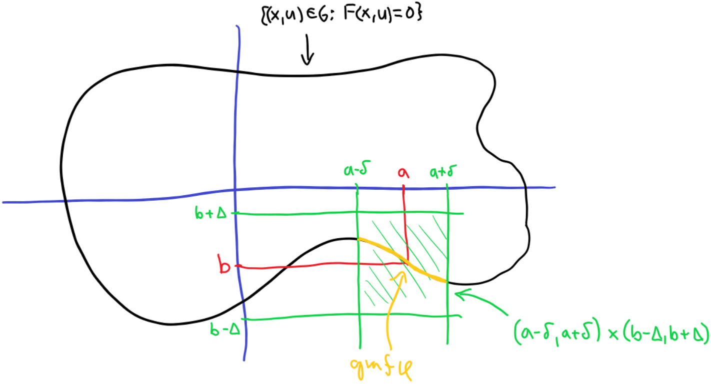
- $F(a, b) = 0$,
- $\frac{\partial F}{\partial u}(a, b) \neq 0$.

Potom existují $\delta, \Delta > 0$ *a* C^k *funkce* $\varphi : (a - \delta, a + \delta) \to \mathbb{R}$*,* $\varphi(a) = b$*, že*

- $F(x, \varphi(x)) = 0$
- $\{(x, u) \in G : F(x, u) = 0\} \cap [(a - \delta, a + \delta) \times (b - \Delta, b + \Delta)] = \text{graf}\ \varphi$

pro $d = m = 1$

- $F(x, \varphi(x)) = 0$
- $\{(x, u) \in G : F(x, u) = 0\} \cap [(a - \delta, a + \delta) \times (b - \Delta, b + \Delta)] = \operatorname{graf} \varphi$



$d = m = 1$

Mějme

- $F(x,u) = u^5 + xu^2 + x + u$, $(G = \mathbb{R}^2)$
- $(a,b) = (0,0)$

Pak

- $F(a,b) = 0$
- $\frac{\partial F}{\partial u}(x,u) = 5u^4 + 2xu + 1$, a tedy $\frac{\partial F}{\partial u}(0,0) = 1 \neq 0$.

Existuje tedy $\delta > 0$ a $\varphi : (-\delta,\delta) \to \mathbb{R}$, $\varphi(0) = 0$, splňující

$$F(x,\varphi(x)) = 0, \quad x \in: (-\delta,\delta).$$

Navíc, protože $F \in C^\infty(\mathbb{R}^2)$, je i $\varphi \in C^\infty((-\delta,\delta))$.
Zkusíme spočítat $\varphi'(x)$. Máme

$$0 = F(x,\varphi(x)) = \varphi(x)^5 + x\varphi(x)^2 + x + \varphi(x)$$

Tedy

$$0 = 5\varphi(x)^4\varphi'(x) + \varphi(x)^2 + 2x\varphi(x)\varphi'(x) + 1 + \varphi'(x)$$

$d = m = 1$

Existuje $\delta > 0$ a $\varphi : (-\delta, \delta) \to \mathbb{R}$, $\varphi(0) = 0$, splňující

$$F(x, \varphi(x)) = 0, \quad x \in: (-\delta, \delta).$$

Navíc, protože $F \in C^\infty(\mathbb{R}^2)$, je i $\varphi \in C^\infty((-\delta, \delta))$.
Zkusíme spočítat $\varphi'(x)$. Máme

$$0 = F(x, \varphi(x)) = \varphi(x)^5 + x\varphi(x)^2 + x + \varphi(x)$$

Tedy

$$0 = 5\varphi(x)^4\varphi'(x) + \varphi(x)^2 + 2x\varphi(x)\varphi'(x) + 1 + \varphi'(x)$$

Po úpravě

$$\varphi'(x) = -\frac{\varphi(x)^2 + 1}{5\varphi(x)^4 + 2x\varphi(x) + 1}$$

Protože $\varphi(0) = 0$ máme $\varphi'(0) = -1$.
Zkusíme spočítat ještě $\varphi''(0)$, máme

$$0 = 20\varphi(x)^3(\varphi'(x))^2 + 5\varphi(x)^4\varphi''(x) + 2\varphi(x)\varphi'(x)$$
$$+ 2\varphi(x)\varphi'(x) + 2x(\varphi'(x))^2 + 2x\varphi(x)\varphi''(x) + \varphi''(x)$$

$d = m = 1$

Máme

$$0 = F(x, \varphi(x)) = \varphi(x)^5 + x\varphi(x)^2 + x + \varphi(x),$$

$$0 = 5\varphi(x)^4\varphi'(x) + \varphi(x)^2 + 2x\varphi(x)\varphi'(x) + 1 + \varphi'(x)$$

Protože $\varphi(0) = 0$ máme $\varphi'(0) = -1$.
Dále

$$\begin{aligned} 0 = {} & 20\varphi(x)^3(\varphi'(x))^2 + 5\varphi(x)^4\varphi''(x) + 2\varphi(x)\varphi'(x) \\ & + 2\varphi(x)\varphi'(x) + 2x(\varphi'(x))^2 + 2x\varphi(x)\varphi''(x) + \varphi''(x) \end{aligned}$$

Tedy (protože $\varphi(0) = 0$ a $\varphi'(0) = -1$)

$$\begin{aligned} 0 = {} & 0 + 0 + 0 \\ & + 0 + 0 + 0 + \varphi''(0) \end{aligned}$$

a $\varphi''(0) = 0$. Platí tedy například $\varphi(x) = -x + o(x^2)$.
Zároveň víme, že φ je rostoucí na okolí bodu 0 (podobně můžeme vyšetřovat konvexitu/konkavitu či existenci extrému v 0).

$d = m = 1$

Máme $F(x,u) = u^5 + xu^2 + x + u$ a $F(x, \varphi(x)) = 0$, $\varphi(0) = 0$, $\varphi'(0) = -1$ a $\varphi''(0) = 0$.$\varphi'(0)$ jsme spočítali ze vzorečku

$$\varphi'(x) = -\frac{\varphi(x)^2 + 1}{5\varphi(x)^4 + 2x\varphi(x) + 1} = -\frac{u^2 + 1}{5u^4 + 2xu + 1} = -\frac{\frac{\partial F}{\partial x}(x,u)}{\frac{\partial F}{\partial u}(x,u)}$$

a tedy $\varphi'(0) = -\frac{\frac{\partial F}{\partial x}(0, \varphi(0))}{\frac{\partial F}{\partial u}(0, \varphi(0))} = -\frac{\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial u}(0,0)}$

Obecný vzoreček odvodíme podobně pomocí řetízkového pravidla

$$F(f(x), g(x)) = \frac{\partial F}{\partial x}(f(x), g(x)) \cdot f'(x) + \frac{\partial F}{\partial u}(f(x), g(x)) \cdot g'(x)$$

použitého pro $f(x) = x$, $g(x) = \varphi(x)$.

Máme $0 = (F(x, \varphi(x)))' = \frac{\partial F}{\partial x}(x, \varphi(x)) + \frac{\partial F}{\partial u}(x, \varphi(x)) \cdot \varphi'(x)$

a stačí použít $\varphi(a) = b$, což dává

$$\varphi'(a) = -\frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial u}(a,b)}$$

$d = 1, m = 1$

Máme $F(x,u) = u^5 + xu^2 + x + u$ a $F(x,\varphi(x)) = 0$, $\varphi(0) = 0$, $\varphi'(0) = -1$ a $\varphi''(0) = 0$. $\varphi'(0)$ jsme spočítali ze vzorečku

$$\varphi'(x) = -\frac{\varphi(x)^2 + 1}{5\varphi(x)^4 + 2x\varphi(x) + 1} = -\frac{u^2+1}{5u^4 + 2xu + 1} = -\frac{\frac{\partial F}{\partial x}(x,u)}{\frac{\partial F}{\partial u}(x,u)}$$

a tedy $\varphi'(0) = -\frac{\frac{\partial F}{\partial x}(0,\varphi(0))}{\frac{\partial F}{\partial u}(0,\varphi(0))} = -\frac{\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial u}(0,0)}$

Obecný vzoreček odvodíme podobně (pomocí řetízkového pravidla)

$F(f(x), g(x)) = \frac{\partial F}{\partial x}(f(x), g(x))f'(x) + \frac{\partial F}{\partial u}(f(x), g(x))g'(x)$

pro $f(x) = x$, $g(x) = \varphi(x)$

máme $0 = (F(x,\varphi(x)))' = \frac{\partial F}{\partial x}(x,\varphi(x)) + \frac{\partial F}{\partial u}(x,\varphi(x))\varphi'(x)$ a stačí použít $\varphi(a) = b$, což dává

$$\varphi'(a) = -\frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial u}(a,b)}$$

pro $d = 2,\ m = 1$

Věta (o implicitní funkci)

Nechť $G \subset \mathbb{R}^3$ *je otevřená,* $F : G \to \mathbb{R}$, $F \in C^k(G)$ *pro nějaké* $k \in \mathbb{N}$, $(a_1, a_2, b) = (a, b) \in G \subset \mathbb{R}^2 \times \mathbb{R}$. *Předpokládejme, že*

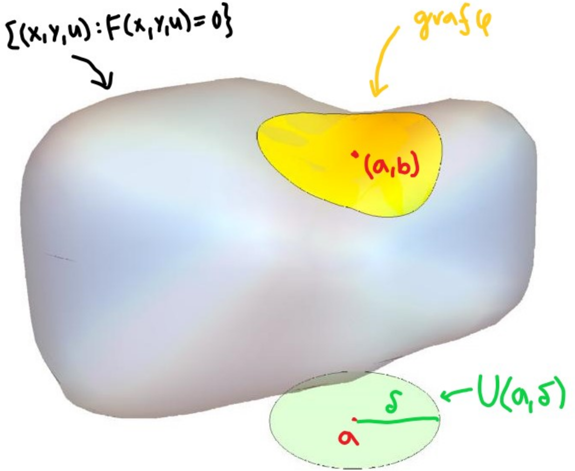
- $F(a, b) = 0$,
- $\frac{\partial F}{\partial u}(a, b) \neq 0$.

Potom existují $\delta, \Delta > 0$ *a* C^k *funkce* $\varphi : U(a, \delta) \to \mathbb{R}$, $\varphi(a) = b$, *že*

- $F(x, y, \varphi(x, y)) = 0$
- $\{(x, y, u) \in G : F(x, y, u) = 0\} \cap [U(a, \delta) \times (b - \Delta, b + \Delta)] = \operatorname{graf} \varphi$

pro $d = 2,\ m = 1$

- $F(x, y, \varphi(x, y)) = 0$
- $\{(x, y, u) \in G : F(x, y, u) = 0\} \cap [U(a, \delta) \times (b - \Delta, b + \Delta)] = \operatorname{graf} \varphi$



$d = 2,\ m = 1$

$F(x,y,u) = \sin(xy) + e^{x^2 u} - u^3,\ a = (0,1),\ b = 1$

- F má spojité parciální derivace všech řádů
- $F(0,1,1) = 0$
- $\frac{\partial F}{\partial u}(x,y,u) = x^2 e^{x^2 u} - 3u^2$ a tedy $\frac{\partial F}{\partial u}(0,1,1) = -3 \neq 0$

Potom existují $\delta, \Delta > 0$ a C^∞ funkce $\varphi : U(a,\delta) \to \mathbb{R}$, že

- $F(x,y,\varphi(x,y)) = 0$
- $\{(x,y,u) \in G : F(x,y,u) = 0\} \cap [U(a,\delta) \times (b-\Delta, b+\Delta)] = \operatorname{graf} \varphi$

Spočítáme $\nabla\varphi(0,1)$, obecně platí

$$\frac{\partial \varphi}{\partial x}(a) = -\frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial u}(a,b)}, \quad \frac{\partial \varphi}{\partial y}(a) = -\frac{\frac{\partial F}{\partial y}(a,b)}{\frac{\partial F}{\partial u}(a,b)}.$$

$$\frac{\partial F}{\partial x}(x,y,u) = y\cos(xy) + 2xue^{x^2 u}, \quad \frac{\partial F}{\partial y}(x,y,u) = x\cos(xy).$$

$$\frac{\partial F}{\partial x}(0,1,1) = 1, \quad \frac{\partial F}{\partial y}(0,1,1) = 0, \quad \frac{\partial F}{\partial u}(0,1,1) = -3.$$

$d = 2, \; m = 1$

$F(x,y,u) = \cos(xy) + e^{y^2 u} - u^3, \; a = (0,1), \; b = 1$
Spočítáme $\nabla\varphi(0,1)$, obecně platí

$$\frac{\partial \varphi}{\partial x}(a) = -\frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial u}(a,b)}, \quad \frac{\partial \varphi}{\partial y}(a) = -\frac{\frac{\partial F}{\partial y}(a,b)}{\frac{\partial F}{\partial u}(a,b)}.$$

$$\frac{\partial F}{\partial x}(0,1,1) = 1, \quad \frac{\partial F}{\partial y}(0,1,1) = 0, \quad \frac{\partial F}{\partial u}(0,1,1) = -3.$$

Tedy $\displaystyle \frac{\partial \varphi}{\partial x}(0,1) = -\frac{\frac{\partial F}{\partial x}(0,1,1)}{\frac{\partial F}{\partial u}(0,1,1)} = \frac{1}{3}, \quad \frac{\partial \varphi}{\partial y}(a) = -\frac{\frac{\partial F}{\partial y}(a,b)}{\frac{\partial F}{\partial u}(a,b)} = 0$

a $\nabla\varphi(0,1) = (\frac{1}{3}, 0)$. Protože φ je C^1 existuje totální diferenciál $d\varphi(0,1) = \frac{1}{3} dx$.

Případ $m = 1, \; d \in \mathbb{N}$, analogicky $\rightarrow \displaystyle \frac{\partial \varphi}{\partial x_i}(a) = -\frac{\frac{\partial F}{\partial x_i}(a,b)}{\frac{\partial F}{\partial u}(a,b)}.$

$d = 1, m = 2$

Věta (o implicitním zobrazení (křivce))

Máme $G \subset \mathbb{R} \times \mathbb{R}^2$ *otevřenou,* $(a_1, b_1, b_2) = (a, b) \in G$, $F : G \to \mathbb{R}^2$,

$$F : (x, u, v) \mapsto (F_1(x, u, v), F_2(x, u, v)).$$

Předpokládejme, že

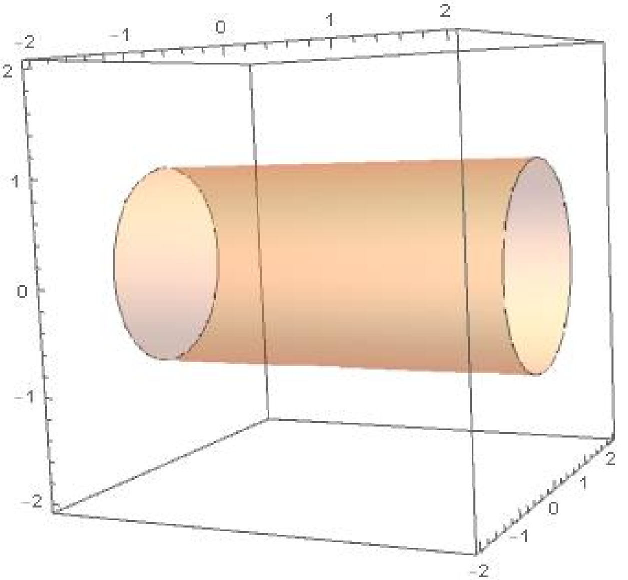
- $F(a, b) = 0$,
- $\det(J_U) := \det \begin{pmatrix} \frac{\partial F_1}{\partial u}(a, b) & \frac{\partial F_1}{\partial v}(a, b) \\ \frac{\partial F_2}{\partial u}(a, b) & \frac{\partial F_2}{\partial v}(a, b) \end{pmatrix} \neq 0.$

Potom existují $\delta, \Delta > 0$ *a* C^k *funkce* $\varphi : U(a, \delta) \to \mathbb{R}$, $\varphi(a) = b$, *že*

- $F(x, \varphi_1(x), \varphi_2(x)) = 0$
- $\{(x, u, v) \in G : F(x, u, v) = 0\} \cap [(a - \delta, a + \delta) \times U(b, \Delta)] = \operatorname{graf} \varphi$

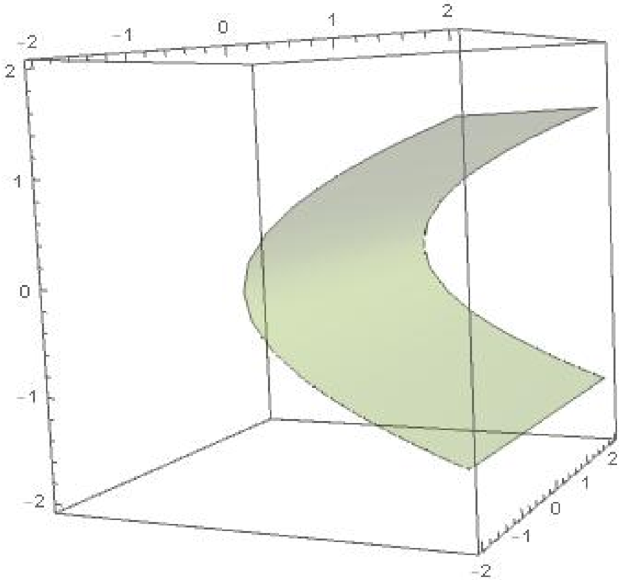
pro $d = 1, \ m = 2$

$\{(x, y, z) : y^2 + z^2 - 1 = 0\}$



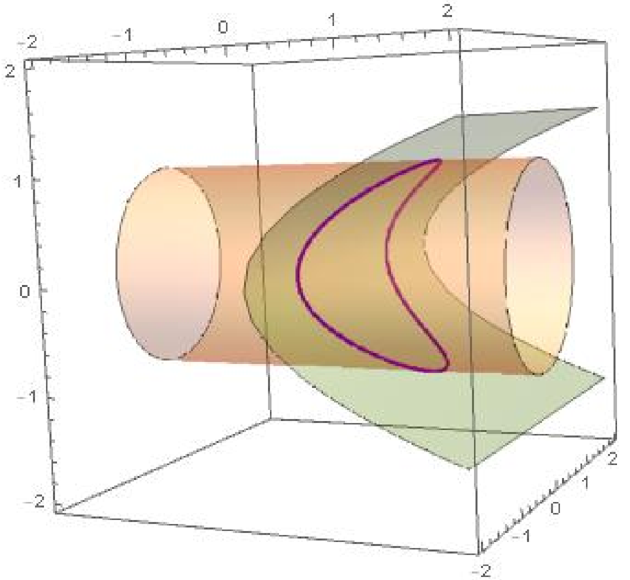
pro $d = 1,\ m = 2$

$$\{(x, y, z) : y^2 - z = 0\}$$



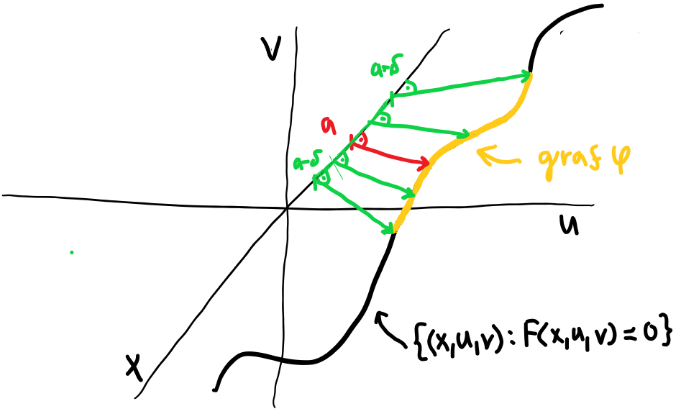
pro $d = 1, \; m = 2$

$$\{(x, y, z) : y^2 + z^2 = 1, \; y^2 - z = 0\}$$



pro $d = 1, \ m = 2$

- $F(x, \varphi_1(x), \varphi_2(x)) = 0$
- $\{(x, u, v) \in G : F(x, u, v) = 0\} \cap [(a - \delta, a + \delta) \times U(b, \Delta)] = \operatorname{graf} \varphi$



$d = 1, m = 2$

Spočítáme $J_\varphi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix}$

Máme

$$
\begin{aligned}
F_1(x, \varphi_1(x), \varphi_2(x)) &= 0 \\
F_2(x, \varphi_1(x), \varphi_2(x)) &= 0
\end{aligned}
$$

Derivováním těchto rovnic podle x dostaneme v bodě (a, b)

$$
\begin{aligned}
\frac{\partial F_1}{\partial x}(a,b) + \frac{\partial F_1}{\partial u}(a,b) \cdot \frac{\partial \varphi_1}{\partial x}(a) + \frac{\partial F_1}{\partial v}(a,b) \cdot \frac{\partial \varphi_2}{\partial x}(a) &= 0 \\
\frac{\partial F_2}{\partial x}(a,b) + \frac{\partial F_2}{\partial u}(a,b) \cdot \frac{\partial \varphi_1}{\partial x}(a) + \frac{\partial F_2}{\partial v}(a,b) \cdot \frac{\partial \varphi_2}{\partial x}(a) &= 0
\end{aligned}
$$

Což lze přepsat jako

$$
\begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b) \end{pmatrix}
$$

$d = 1, m = 2$

Máme

$$\begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b) \end{pmatrix}$$

a tedy

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b), \end{pmatrix}$$

Poznamenejme, že J_U^{-1} existuje protože $\det(J_U) \neq 0$.
Pro obecné m analogicky:

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \vdots \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial u_1}(a,b) & \cdots & \frac{\partial F_1}{\partial u_m}(a,b) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial u_1}(a,b) & \cdots & \frac{\partial F_m}{\partial u_m}(a,b) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \vdots \\ \frac{\partial F_2}{\partial x}(a,b), \end{pmatrix}$$

$d = 1, m = 2$

Mějme $(0,0,1) \in \mathbb{R} \times \mathbb{R}^2$, tj. $a = 0$, $b = (0,1)$ a

$$F_1(x,u,v) = x^4 + u^4 + v^4 - 1,$$
$$F_1(x,u,v) = xuv + e^{x+u+v} - e$$

Zjevně $F_1(0,0,1) = F_2(0,0,1) = 0$. Dále

$$J_F = (J_X | J_U) = \left(\begin{array}{c|cc} 4x^3 & 4u^3 & 4v^3 \\ uv + e^{x+u+v} & xv + e^{x+u+v} & xu + e^{x+u+v} \end{array}\right)$$

a tedy

$$J_F(0,0,1) = \left(\begin{array}{c|cc} 0 & 0 & 4 \\ e & e & e \end{array}\right)$$

$$\det \begin{pmatrix} 0 & 4 \\ e & e \end{pmatrix} = -4e \neq 0, \quad \begin{pmatrix} 0 & 4 \\ e & e \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{e} & \frac{1}{e} \\ \frac{1}{4} & 0 \end{pmatrix}$$

Podle vzorečku pak máme

$$J_\varphi(0,1) = -J_U^{-1} J_X = -\begin{pmatrix} -\frac{1}{e} & \frac{1}{e} \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

d=m=2

Máme $G \subset \mathbb{R}^2 \times \mathbb{R}^2$ otevřenou, $(a_1, a_2, b_1, b_2) = (a, b) \in G$, $F : G \to \mathbb{R}^2$,

$$F : (x, y, u, v) \mapsto (F_1(x, y, u, v), F_2(x, y, u, v)).$$

Předpokládejme, že

- $F(a, b) = 0$,
- $\det \begin{pmatrix} \frac{\partial F_1}{\partial u}(a, b) & \frac{\partial F_1}{\partial v}(a, b) \\ \frac{\partial F_2}{\partial u}(a, b) & \frac{\partial F_2}{\partial v}(a, b) \end{pmatrix} \neq 0.$

Potom existují $\delta, \Delta > 0$ a C^k funkce $\varphi : U(a, \delta) \to \mathbb{R}$, $\varphi(a) = b$, že

- $F(x, y, \varphi_1(x, y), \varphi_2(x, y)) = 0$
- $\{(x, y, u, v) \in G : F(x, y, u, v) = 0\} \cap [U(a, \delta) \times U(b, \Delta)] = \text{graf}\, \varphi$

d=m=2

Spočítáme $J_\varphi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) & \frac{\partial \varphi_1}{\partial y}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) & \frac{\partial \varphi_2}{\partial y}(a) \end{pmatrix}$

Máme

$$\begin{aligned} F_1(x, y, \varphi_1(x, y), \varphi_2(x, y)) &= 0 \\ F_2(x, y, \varphi_1(x, y), \varphi_2(x, y)) &= 0 \end{aligned}$$

Derivováním těchto rovnic podle x dostaneme v bodě (a, b)

$$\begin{aligned} \frac{\partial F_1}{\partial x}(a, b) + \frac{\partial F_1}{\partial u}(a, b) \cdot \frac{\partial \varphi_1}{\partial x}(a) + \frac{\partial F_1}{\partial v}(a, b) \cdot \frac{\partial \varphi_2}{\partial x}(a) &= 0 \\ \frac{\partial F_2}{\partial x}(a, b) + \frac{\partial F_2}{\partial u}(a, b) \cdot \frac{\partial \varphi_1}{\partial x}(a) + \frac{\partial F_2}{\partial v}(a, b) \cdot \frac{\partial \varphi_2}{\partial x}(a) &= 0 \end{aligned}$$

Což lze přepsat jako

$$\begin{pmatrix} \frac{\partial F_1}{\partial u}(a, b) & \frac{\partial F_1}{\partial v}(a, b) \\ \frac{\partial F_2}{\partial u}(a, b) & \frac{\partial F_2}{\partial v}(a, b) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial x}(a, b) \\ \frac{\partial F_2}{\partial x}(a, b) \end{pmatrix}$$

d=m=2

Máme

$$\begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b) \end{pmatrix}$$

Derivováním rovnic podle y dostaneme v bodě (a, b)

$$\frac{\partial F_1}{\partial y}(a,b) + \frac{\partial F_1}{\partial u}(a,b) \cdot \frac{\partial \varphi_1}{\partial y}(a) + \frac{\partial F_1}{\partial v}(a,b) \cdot \frac{\partial \varphi_2}{\partial y}(a) = 0$$
$$\frac{\partial F_2}{\partial y}(a,b) + \frac{\partial F_2}{\partial u}(a,b) \cdot \frac{\partial \varphi_1}{\partial y}(a) + \frac{\partial F_2}{\partial v}(a,b) \cdot \frac{\partial \varphi_2}{\partial y}(a) = 0$$

Což lze přepsat jako

$$\begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial y}(a) \\ \frac{\partial \varphi_2}{\partial y}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial y}(a,b) \\ \frac{\partial F_2}{\partial y}(a,b) \end{pmatrix}$$

d=m=2

Máme

$$\begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial y}(a) \\ \frac{\partial \varphi_2}{\partial y}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial y}(a,b) \\ \frac{\partial F_2}{\partial y}(a,b) \end{pmatrix}$$

a tedy

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial y}(a) \\ \frac{\partial \varphi_2}{\partial y}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial y}(a,b) \\ \frac{\partial F_2}{\partial y}(a,b) \end{pmatrix}$$

d=m=2

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(a) \\ \frac{\partial \varphi_2}{\partial x}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial y}(a) \\ \frac{\partial \varphi_2}{\partial y}(a) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial y}(a,b) \\ \frac{\partial F_2}{\partial y}(a,b) \end{pmatrix}$$

Tedy $J_\varphi = J_U^{-1} J_X$, kde

$$J_X = \begin{pmatrix} \frac{\partial F_1}{\partial x}(a,b) & \frac{\partial F_1}{\partial y}(a,b) \\ \frac{\partial F_2}{\partial x}(a,b) & \frac{\partial F_2}{\partial y}(a,b) \end{pmatrix}, \quad J_U = \begin{pmatrix} \frac{\partial F_1}{\partial u}(a,b) & \frac{\partial F_1}{\partial v}(a,b) \\ \frac{\partial F_2}{\partial u}(a,b) & \frac{\partial F_2}{\partial v}(a,b) \end{pmatrix}$$

d=m=2

Ukážeme, že soustava

$$xu^2 + e^{v^2+y} - xy + v = 2,$$
$$v^2 + x^3y + \frac{u}{x} - y^3 = 1,$$

určuje na okolí bodu $(1, 0, 1, 0)$ implicitně zadané zobrazení φ (proměnných x a y) a spočítáme $J_\varphi(1, 0)$.

Položíme

$$F_1(x, y, u, v) = xu^2 + e^{v^2+y} - xy + v - 2,$$
$$F_1(x, y, u, v) = v^2 + x^3y + \frac{u}{x} - y^3 - 1.$$

Zjevně $F_1(1, 0, 1, 0) = F_2(1, 0, 1, 0) = 0$. Dále

$$J_F = (J_X|J_U) = \left(\begin{array}{cc|cc} u^2 - y & e^{v^2+y} - x & 2xu & 2ve^{v^2+y} + 1 \\ 3x^2y - \frac{u}{x^2} & x^3 - 3y^2 & \frac{1}{x} & 2v \end{array}\right)$$

a tedy

$$J_F(1, 0, 1, 0) = \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ -1 & 1 & 1 & 0 \end{array}\right)$$

d=m=2

$$F_1(x,y,u,v) = xu^2 + e^{v^2+y} - xy + v - 2,$$
$$F_1(x,y,u,v) = v^2 + x^3y + \frac{u}{x} - y^3 - 1.$$

Zjevně $F_1(1,0,1,0) = F_2(1,0,1,0) = 0$ a

$$J_F(1,0,1,0) = (J_X|J_U) = \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ -1 & 1 & 1 & 0 \end{array}\right)$$

$$\det(J_U) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1, \qquad \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

Existují tedy $\delta, \Delta > 0$ a $\varphi = (\varphi_1, \varphi_2) : U((1,0),\delta) \to \mathbb{R}^2$, že $\varphi(1,0) = (1,0)$ a

$$F(x,y,\varphi(x,y),\varphi(x,y)) = (0,0)$$

Podle vzorečku pak máme

$$J_\varphi(1,0) = -J_U^{-1}J_X = -\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}.$$

Věta (o implicitním zobrazení)

Nechť $G \subset \mathbb{R}^{d+m}$ je otevřená, $F : G \to \mathbb{R}^m$, $F \in C^k(G)$ pro nějaké $k \in \mathbb{N}$, $(a, b) \in G \subset \mathbb{R}^{d+m}$. Předpokládejme, že

- $F(a, b) = 0$,
- $\det(J_U) \neq 0$.

Potom existují $\delta, \Delta > 0$ taková, že pro každé $x \in U(a, \delta)$ existuje právě jedno $y \in U(b, \Delta)$ takové, že $F(x, u) = 0$. Navíc, označíme-li jako φ zobrazení přiřazující bodu $x \in U(a, \delta)$ bod $y \in U(b, \Delta)$ jako výše, je toto zobrazení $C^k(U(a, \delta))$.
Zároveň platí $J_\varphi = -J_U^{-1} J_X$.

Pro připomenutí:

$$J_X = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(a,b) & \cdots & \frac{\partial F_1}{\partial x_d}(a,b) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x}(a,b) & \cdots & \frac{\partial F_m}{\partial x_d}(a,b) \end{pmatrix}, \quad J_U = \begin{pmatrix} \frac{\partial F_1}{\partial u_1}(a,b) & \cdots & \frac{\partial F_1}{\partial u_m}(a,b) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial u_1}(a,b) & \cdots & \frac{\partial F_m}{\partial u_m}(a,b) \end{pmatrix}$$

inverzní zobrazení ($d = m$)

Mějme soustavu

$$f_1 u_1, \dots, u_d) = x_1$$

$$\vdots$$

$$f_d(u_1, \dots, u_d) = x_d$$

Pro $f_i \in C^k(G)$, $G \subset \mathbb{R}^d$ otevřená, $i = 1, \dots d$. Položme $f = (f_1, \dots, f_d)$ a $F = (F_1, \dots, F_d)$, kde

$$F_i(x_1, \dots, x_d, u_1, \dots, u_d) = f_1(u_1, \dots, u_d) - x_i, \quad i = 1, \dots, d.$$

Potom $f : G \to \mathbb{R}^d$ a $F : G \times \mathbb{R}^d \to \mathbb{R}^d$ jsou C^k a

$$J_F(x, u) = (-\operatorname{Id}_d | J_f(u))$$

Pokud $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d$ je nějakým řešením soustavy $F(a, b) = 0$ a $\det(J_f(b)) \neq 0$ potom na okolí bodu a lze definovat zobrazení φ, pro které platí $F(x, \varphi(x)) = 0$, což je totéž jako

$$f(\varphi(x)) = \operatorname{Id}_d \cdot x$$

Tedy φ je inverzní zobrazení (na okolí bodu a) k zobrazení f.
Navíc $J_f = -J_f^{-1}(-\operatorname{Id}_d) = J_f^{-1}$.